

Linear Differential Equations with Constant Coefficients

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Linear Differential Equations

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the n -th order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = X,$$

where $p_1, p_2, \dots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X$$

where $k_1, k_2, \dots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

Theorem I

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0 \dots (1)$$

then $c_1 y_1 + c_2 y_2 (= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1),

$$\therefore \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \cdots + k_n y_1 = 0 \dots (2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \cdots + k_n y_2 = 0 \dots (3)$$

Theorem II

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n(c_1y_1 + c_2y_2)}{dx^n} + k_1 \frac{d^{n-1}(c_1y_1 + c_2y_2)}{dx^{n-1}} + \cdots + k_n(c_1y_1 + c_2y_2)$$

$$\begin{aligned} \text{i.e.,} \quad &= c_1 \left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + k_n y_1 \right) + c_2 \left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + k_n y_2 \right) \\ &= c_1(0) + c_2(0) = 0 \end{aligned}$$

$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + k_n u = 0 \quad (4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the n th order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \dots, y_n$ are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \cdots + c_ny_n (= u)$ is its complete solution.

Theorem III

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_n y = X \dots (5)$$

then

$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + k_n v = X \dots (6)$$

Adding (4) and (6), we have

$$\frac{d^n (u + v)}{dx^n} + k_1 \frac{d^{n-1} (u + v)}{dx^{n-1}} + \cdots + k_n (u + v) = X$$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)**, and the part v is called the **particular integral (P.I.)** of (5).

$\therefore$ the complete solution (C.S.) of (5) is $y = \text{C.F.} + \text{P.I.}$

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e., a particular solution of (5).

Operator D

Denoting

$$\frac{d}{dx}, \quad \frac{d^2}{dx^2}, \quad \frac{d^3}{dx^3}, \quad \text{etc.}$$

by D, D^2, D^3 , etc., so that

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \quad \frac{d^3y}{dx^3} = D^3y,$$

etc., the equation (5) above can be written in the symbolic form

$$(D^n + k_1 D^{n-1} + \cdots + k_n)y = X,$$

i.e.,

$$f(D)y = X,$$

where

$$f(D) = D^n + k_1 D^{n-1} + \cdots + k_n,$$

i.e., a polynomial in D .

Thus, the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity, i.e., $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order.

For instance,

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 3y = (D^2 + 2D - 3)y$$

$$= (D + 3)(D - 1)y$$

or

$$= (D - 1)(D + 3)y.$$

Rules for Finding the Complementary Function

To solve the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0,$$

where k 's are constants.

The equation (1) in symbolic form is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = 0.$$

Its symbolic co-efficient equated to zero, i.e.,
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$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n = 0$$

is called the *auxiliary equation (A.E.)*. Let $m_1, m_2, \dots, m_n$ be its roots.

Case I. If all the roots are real and different, then (2) is equivalent to

$$(D - m_1)(D - m_2) \cdots (D - m_n)y = 0.$$

Now (3) will be satisfied by the solution of

$$(D - m_n)y = 0,$$

i.e.,

$$\frac{dy}{dx} - m_n y = 0.$$

This is a Leibnitz's linear equation and its integrating factor (I.F.) is

$$e^{-m_n x}.$$

Therefore, its solution is

$$ye^{-m_n x} = c_n,$$

i.e.,

$$y = c_n e^{m_n x}.$$

Similarly, since the factors can be taken in any order, it will be satisfied by the solutions of

$$(D - m_1)y = 0, \quad (D - m_2)y = 0, \quad \dots$$

i.e.,

$$y = c_1 e^{m_1 x}, \quad y = c_2 e^{m_2 x}, \quad \dots$$

Thus, the complete solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}. \quad (4)$$

Case II. If two roots are equal (i.e., $m_1 = m_2$), then (4) becomes

$$y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

Since $c_1 + c_2$ is one arbitrary constant, we get

$$y = Ce^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows.

The part of the complete solution corresponding to the repeated root is the complete solution of

$$(D - m_1)^2 y = 0.$$

Putting

$$(D - m_1)y = z,$$

it becomes

$$(D - m_1)z = 0,$$

or

$$\frac{dz}{dx} - m_1 z = 0.$$

This is a Leibnitz's linear equation in z , and its integrating factor is

$$e^{-m_1 x}.$$

Therefore, its solution is

$$ze^{-m_1 x} = c_1,$$

or

$$z = c_1 e^{m_1 x}.$$

Thus,

$$(D - m_1)y = z = c_1 e^{m_1 x},$$

or

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}. \quad (5)$$

Its integrating factor being e^{-m_1x} , the solution of (5) is

$$ye^{-m_1x} = \int c_1 dx + c_2.$$

Hence,

$$y = (c_1x + c_2)e^{m_1x}.$$

Thus, the complete solution of (1) is

$$y = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

If, however, the auxiliary equation has three equal roots (i.e., $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + \cdots + c_ne^{m_nx}.$$

Case III. If one pair of roots is imaginary, i.e.,

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta,$$

then the complete solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} + c_3 e^{m_3x} + \cdots + c_n e^{m_nx}.$$

Taking $e^{\alpha x}$ common, we get

$$y = e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3x} + \cdots + c_n e^{m_nx}.$$

By Euler's theorem,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

we obtain

$$\begin{aligned} y = e^{\alpha x} [& c_1 (\cos \beta x + i \sin \beta x) \\ & + c_2 (\cos \beta x - i \sin \beta x)] \\ & + c_3 e^{m_3x} + \cdots + c_n e^{m_nx}. \end{aligned}$$

Therefore,

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x},$$

where

$$C_1 = c_1 + c_2, \quad C_2 = i(c_1 - c_2).$$

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Case IV. If two pairs of imaginary roots are equal, i.e.,

$$m_1 = m_2 = \alpha + i\beta, \quad m_3 = m_4 = \alpha - i\beta,$$

then, by Case II, the complete solution is

$$y = e^{\alpha x} \left[(c_1 x + c_2) \cos \beta x \right. \\ \left. + (c_3 x + c_4) \sin \beta x \right] \\ + \cdots + c_n e^{m_n x}.$$

Examples: Linear Differential Equations

Example 1 Solve

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0,$$

given $x(0) = 0$, $\frac{dx}{dt}(0) = 15$.

Solution. Given equation in symbolic form is

$$(D^2 + 5D + 6)x = 0.$$

Its A.E. is

$$D^2 + 5D + 6 = 0,$$

i.e.,

$$(D + 2)(D + 3) = 0 \Rightarrow D = -2, -3.$$

Therefore, the C.S. is

$$x = c_1 e^{-2t} + c_2 e^{-3t}$$

and

$$\frac{dx}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

When $t = 0$, $x = 0$, we get

$$0 = c_1 + c_2. \quad (i)$$

When $t = 0$, $\frac{dx}{dt} = 15$, we get

$$15 = -2c_1 - 3c_2. \quad (ii)$$

Solving (i) and (ii),

$$c_1 = 15, \quad c_2 = -15.$$

Hence, the required solution is

$$x = 15(e^{-2t} - e^{-3t}).$$

Example 2 Solve

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0.$$

Solution. Given equation in symbolic form is

$$(D^2 + 6D + 9)x = 0.$$

The A.E. is

$$D^2 + 6D + 9 = 0,$$

i.e.,

$$(D + 3)^2 = 0 \Rightarrow D = -3, -3.$$

Hence, the C.S. is

$$x = (c_1 + c_2 t)e^{-3t}.$$

Example 3 Solve

$$(D^3 + D^2 + 4D + 4)y = 0.$$

Solution. Here the A.E. is

$$D^3 + D^2 + 4D + 4 = 0,$$

i.e.,

$$(D^2 + 4)(D + 1) = 0.$$

Therefore,

$$D = -1, \pm 2i.$$

Hence, the C.S. is

$$y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x.$$

Example 4 Solve

$$(i) \quad (D^2 - 4D + 4)y = 0$$

$$(ii) \quad (D^2 + 1)^3 y = 0, \quad \text{where } D = \frac{d}{dx}.$$

Solution.

(i) The A.E. is

$$D^2 - 4D + 4 = 0$$

or

$$(D^2 - 2)^2 = 0.$$

Thus,

$$D^2 = 2, \quad 2,$$

i.e.,

$$D = \pm\sqrt{2}, \pm\sqrt{2}.$$

Hence, the C.S. is

$$y = (c_1 + c_2x)e^{\sqrt{2}x} + (c_3 + c_4x)e^{-\sqrt{2}x}.$$

(ii) The A.E. is

$$(D^2 + 1)^3 = 0.$$

Therefore,

$$D = \pm i, \pm i, \pm i.$$

Hence, the C.S. is

$$y = (c_1 + c_2x + c_3x^2) \cos x + (c_4 + c_5x + c_6x^2) \sin x.$$

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Example 5 Solve

$$\frac{d^4 x}{dt^4} + 4x = 0.$$

Solution. Given equation in symbolic form is

$$(D^4 + 4)x = 0.$$

The auxiliary equation (A.E.) is

$$D^4 + 4 = 0,$$

or

$$(D^2 + 2)^2 - (2D)^2 = 0.$$

Therefore,

$$(D^2 + 2D + 2)(D^2 - 2D + 2) = 0.$$

Either

$$D^2 + 2D + 2 = 0$$

or

$$D^2 - 2D + 2 = 0.$$

Hence,

$$D = \frac{-2 \pm \sqrt{4 - 8}}{2} \quad \text{and} \quad D = \frac{2 \pm \sqrt{4 - 8}}{2}.$$

That is,

$$D = -1 \pm i \quad \text{and} \quad D = 1 \pm i.$$

Hence, the required solution is

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$$x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t).$$